Int. J. Heat Mass Transfer. Vol. 27, No. 8, pp. 1414–1417, 1984
Printed in Great Britain

0017–9310/84 $3.00 + 0.00

PERIODIC SOLUTIONS OF HEAT TRANSFER FOR FLOW THROUGH A PERIODIC ASSEMBLAGE OF SPHERES

REUVEN TAL (THAU),*† DAH NAIN LEE‡ and WILLIAM A. SIRIGNANO*

* Department of Mechanical Engineering, and ‡ Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.

(*Received* 9 *August* 1982 *and in revised form* 9 *November* 1983)

NOMENCLATURE

A	system matrix
A	minor system matrix
a	radius of sphere in assemblage
B	matrix appearing in equation (32), consisting of values or boundaries
b	distance between spheres in assemblage
b	non-homogeneous term in minor system equation
N	number of modules in cylindrical cell
NIT	number of iterated points in finite-difference formulation
Nu	Nusselt number
Pr	Prandtl number
Re	Reynolds number, $2U_\infty a/\nu$
T	dimensionless temperature
T'	temperature
T'_∞	inlet temperature
T'_s	surface temperature
$\bar{T}(x)$	cross-sectional average temperature
T	temperature vector, defined in equation (20)
T	reduced temperature vector
u	velocity component in the x-direction
U_∞	uniform inlet velocity
x	axial cylindrical polar coordinate
y	radial cylindrical polar coordinate
$Y_{\max}$	maximum value of Y in periodic formulation
$Y_{\min}$	minimum value of Y in periodic formulation.

Greek symbols

α	temperature decay factor, defined in equation (10)
ε	error criterion
θ	scaled temperature
λ	eigenfunction defined in equation (7)
ξ	dummy variable used for integration
ψ	streamfunction.

THE PROBLEM of fully-developed flow and heat transfer in ducts having streamwise periodic variations of cross-sectional area has been given recent attention [1–6]. The problem has practical importance in systems such as offset-plane heat exchangers, staggered transverse plate heat exchangers and circular tubes with annular rings inserted for heat transfer augmentation. Tal *et al.* [7] considered the flow field and heat transfer in an array of spheres (as part of a model for interactions between vaporizing fuel droplets) and indicated the periodic nature of the flow field as well as of the scaled temperature function. In the present work, solution procedures are presented for constant wall temperature. For the sake of clarity the solution procedure is presented for a specific geometry but it can be applied to other problems with similar physical conditions. An array of equidistant, equisize spheres is represented by a multisphere cylindrical cell model. In this model, several spheres in tandem are enclosed in a cylindrical duct (Fig. 1). The flow field in this duct has been found to be periodic and has been solved for a module of the cell using a procedure similar to that suggested by Patankar *et al.* [5], however, using streamfunction–vorticity and not primitive variables. The flow field is, therefore, assumed to be known and for the case of incompressible fluid with constant viscosity it will not be coupled to the temperature solution. The main purpose for seeking periodic solutions for the temperature field is to reduce the considerable computer time and storage required for the solution of the multisphere problem. Entrance conditions are not analyzed and whenever we subsequently refer to the first sphere, the reference is to the first in a series of modules in the fully-developed regime.

The energy equation for cylindrical polar coordinates is

$$\frac{Re\,Pr}{2y}\left[\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right]-\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{1}{y}\frac{\partial T}{\partial y}\right]=0, \tag{1}$$

where T is a dimensionless temperature defined as

$$T(x,y)=\frac{T'(x,y)-T'_{\text{ref}}}{T'_\infty-T'_{\text{ref}}}. \tag{2}$$

The boundary conditions on the cylindrical cell envelope and on the axis of symmetry are

$$\frac{\partial T}{\partial y}=0 \quad \text{at} \quad y=y_{\max}, \tag{3}$$

$$\frac{\partial T}{\partial y}=0 \quad \text{at} \quad y=0. \tag{4}$$

As the Navier–Stokes equations under the assumption of constant property values are decoupled from the energy equation, the streamfunction, ψ, can be calculated numerically independently of the energy equation.

We consider the case when the surface temperature of each of the spheres is identical and uniform over the sphere. A problem with similar boundary conditions has been analyzed by Patankar *et al.* [5], but we apply a different procedure which is simpler and more convenient. The surface temperature of the spheres is taken as the reference in equation (2).

PERIODICITY CONDITIONS

The analysis discussed in this section is not directly used in our numerical solution procedure; however, it is essential in justifying one of our assumptions, namely equation (16).

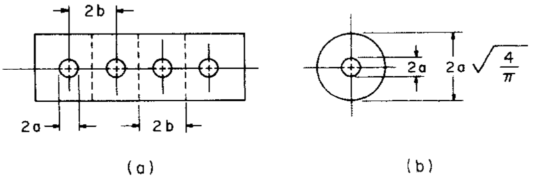

FIG. 1. Cylindrical cell geometry.

---

† Present address: IMI, P.O. Box 32364, Tel Aviv 61230, Israel.

A flow cross-sectional average temperature is defined as

$$T(x)-T_s = \int_{Y_{\min}}^{Y_{\max}} [T(x,y)-T_s]|u|y\,\mathrm{d}y \Big/ \int_{Y_{\min}}^{Y_{\max}} |u|y\,\mathrm{d}y. \quad (5)$$

Based on equation (2) T_s is zero and will be dropped subsequently. A scaled temperature function is defined as

$$\theta(x,y) = \frac{T(x,y)}{T(x)}. \quad (6)$$

where $\theta(x,y)$ is assumed to be a periodic function in the streamwise direction. (This assumption has been justified by Tal *et al.* [7] using a multisphere cylindrical cell model and will be consistent with the following procedure.) We also define a decay function of the cross-sectional average temperature as

$$\lambda(x) = \frac{\mathrm{d}T(x)/\mathrm{d}x}{T(x)} = \mathrm{d}\ln T(x)/\mathrm{d}x. \quad (7)$$

The energy equation is expressed in terms of $\theta(x,y)$ using equations (6) and (7), resulting in the following expression

$$\frac{Re\,Pr}{2y}\left[\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right]-\left[\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{1}{y}\frac{\partial\theta}{\partial y}\right]$$
$$=\theta\left[\lambda^2(x)+\frac{\mathrm{d}\lambda(x)}{\mathrm{d}x}\right]+\left[2\frac{\partial\theta}{\partial x}-\frac{Re\,Pr}{2y}\frac{\partial\psi}{\partial y}\theta\right]\lambda(x). \quad (8)$$

As the LHS of equation (8), containing the periodic functions ψ and θ, is periodic, the RHS has to be periodic as well. This implies the periodicity of $\lambda(x)$. As the length of each module of the assemblage is $2b/a$, we have

$$\psi(x,y) = \psi(x+2b/a,y) = \psi(x+4b/a,y) = \ldots \quad (9)$$
$$\theta(x,y) = \theta(x+2b/a,y) = \theta(x+4b/a,y) = \ldots \quad (10)$$
$$\lambda(x) = \lambda(x+2b/a) = \lambda(x+4b/a) = \ldots \quad (11)$$

As $\lambda(x)$ is an unknown function, it has to be calculated in addition to the function $\theta(x,y)$. Therefore, an additional equation is required and readily given by combining the definitions of θ and $T(x)$ [equations (5) and (6)]

$$\frac{\displaystyle\int_{Y_{\min}}^{Y_{\max}} \theta(x,y)|u|y\,\mathrm{d}y}{\displaystyle\int_{Y_{\min}}^{Y_{\max}} |u|y\,\mathrm{d}y} = 1. \quad (12)$$

One numerical procedure to calculate two functions, $\theta(x,y)$ and $\lambda(x)$, is outlined by Patankar *et al.* [5]. Based upon the periodicity of $\lambda(x)$, we suggest a different procedure.

SUGGESTED SOLUTION PROCEDURE

Due to the periodicity of $\lambda(x)$ and $\theta(x,y)$ we have

$$T(x+N2b/a,y) = T(0)\theta(x,y)\exp\left[\int_0^{x+2Nb/a}\lambda(\xi)\,\mathrm{d}\xi\right]$$
$$= T(0)\theta(x,y)\exp\left[\int_0^{x}\lambda(\xi)\,\mathrm{d}\xi\right]\exp\left[\int_x^{x+2Nb/a}\lambda(\xi)\,\mathrm{d}\xi\right]$$
$$= T(0)\theta(x,y)\exp\left[\int_0^{x}\lambda(\xi)\,\mathrm{d}\xi\right]\exp\left[N\int_0^{2b/a}\lambda(\xi)\,\mathrm{d}\xi\right]$$
$$= T(x,y)\exp\left[N\lambda(\xi)\,\mathrm{d}\xi\right], \quad (13)$$

where N is any integer number of unit cells between two cross-sections.

We define

$$\alpha = \exp\left[\int_0^{2b/a}\lambda(\xi)\,\mathrm{d}\xi\right], \quad (14)$$

and equation (13) now becomes

$$T(x+N2b/a,y) = T(x,y)\alpha^N. \quad (15)$$

For any single unit cell, we have

$$T(x+2b/a,y) = \alpha T(x,y), \quad (16)$$

where α is the decay factor of the temperature across one unit cell, and as heat is transferred from the hot fluid to the cold spheres, we have

$$0<\alpha<1. \quad (17)$$

As $\lambda(x)$ is unknown, so is α.

We now return to the energy equation, equation (1). For any unit cell, we have the boundary conditions (3) and (4). At the cell inlet and outlet, the unknown temperatures are related by the matching condition (16). Note that equation (13) could be differentiated to yield another matching condition on the first derivative of temperature with respect to x.

The finite-difference form of equation (1) using the above conditions results in a *homogeneous* linear system of the form

$$A(\alpha)T = 0. \quad (18)$$

This system has a non-trivial solution only when

$$\det\,[A(\alpha)] = 0. \quad (19)$$

$A(\alpha)$ is typically a 1000×1000 matrix and solving its determinant as a function of α is not feasible. We therefore take a different approach.

NIT is the total number of iterated points. Therefore, there are NIT equations in system (18). The column matrix T has the following form

$$T = \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ \vdots \\ T_{I_{\max},J_{\max}-2} \\ T_{I_{\max},J_{\max}-1} \\ T_{I_{\max},J_{\max}} \end{bmatrix}. \quad (20)$$

We normalize system (18) by $T_{I_{\max},J_{\max}}$ and define a column matrix T by dividing the terms of T by $T_{I_{\max},J_{\max}}$ and dropping the last term (which is obviously 1). The dimension of matrix T is $(NIT-1)\times1$. System (18) is solved using the following procedure. A non-homogeneous system of $NIT-1$ equations, of the form

$$A(\alpha)T = b, \quad (21)$$

is created, by normalizing the first $NIT-1$ equations in system (18) with respect to $T_{I_{\max},J_{\max}}$. [$A(\alpha)$ being the minor of the NIT, NIT term of $A(\alpha)$ and the non-homogeneous term b resulting from the coefficients of $T_{I_{\max},J_{\max}}$ appearing in the last column in the $A(\alpha)$ matrix.]

This system admits a solution for *any* α as long as $A(\alpha)$ is nonsingular.

The last equation of system (18) after being normalized with respect to $T_{I_{\max},J_{\max}}$ has the functional form

$$F(\alpha,T) = 0. \quad (22)$$

As system (21) has $NIT-1$ equations for any specified α it gives a solution for T, so that equation (22) is not satisfied in general. Only for a specific α, are both system (21) and equation (22) satisfied, and this is exactly the solution of equation (19).

The solution procedure suggested is as follows:

Step (1) Specify α.
Step (2) Solve system (21) for T (a reduced system method with conjugate gradient acceleration detailed in Hageman and Young [8] is used for this purpose).
Step (3) Estimate $\varepsilon = F(\alpha, T)$.
Step (4) Repeat Steps (1)–(3) until $\varepsilon = 0$ or in numerical expression $\varepsilon > \varepsilon_{\min}$ ($\varepsilon_{\min}$ is set to 10^{-6}).

In order to obtain a good convergence of the process, a

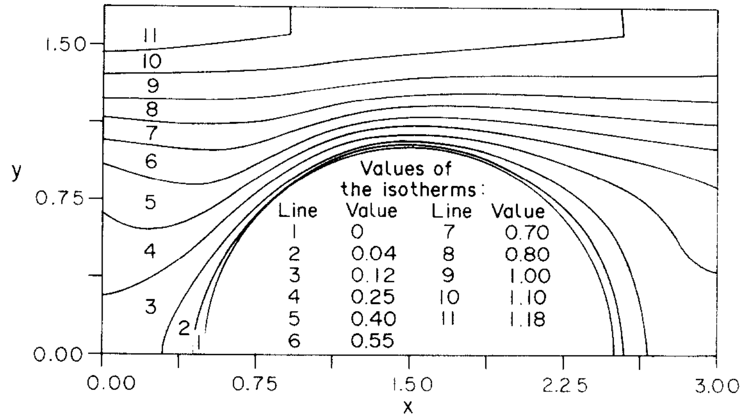

FIG. 2. Isotherm patterns ($Re = 100$, $Pr = 1$, $b/a = 1.5$).

bisection method has been used in Step (4). Convergence is typically obtained in 4–5 iterations (depending on the initial guess of α).

After obtaining a solution for T, a scaling factor, namely $T_{l_{\max}, Y_{\max}}$ is yet to be found. This factor is found easily by specifying an inlet condition

$$T(0) = \frac{\int_{Y_{\min}}^{Y_{\max}} T(0, y)|u|y \, \mathrm{d}y}{\int_{Y_{\min}}^{Y_{\max}} |u|y \, \mathrm{d}y} = 1. \tag{23}$$

The solution for $Re = 100$, $b/a = 1.5$ and $Pr = 1$ is given in Fig. 2. The value of α satisfying equation (19) and system (22) is 0.8738. The average Nusselt number on the sphere, based on the difference between the inlet and surface temperature is 5.935. Using the multisphere cylindrical cell model given in detail by Tal *et al.* [7], the average Nusselt number of a sphere in the bulk of the assemblage, based on local inlet conditions is 5.911. Thus, the agreement of the results is quite satisfactory.

A one-dimensional approximation (neglecting cross-stream variations of the temperature as well as diffusion) presented in the Appendix, yields important results regarding the periodicity of the problem; these results are in good agreement with the analysis as well as the numerical work taking into account the complete energy equation.

CONCLUSIONS

The procedure outlined above is more straightforward than previous procedures for solving periodic problems. No source terms are introduced, no cross-sectional averaging is required and the convergence is quite fast.

REFERENCES

1. B. E. Launder and X. X. Massey, Viscous flow and heat transfer in tube banks, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **100**, 565–571 (1978).
2. D. Ryan, R. G. Carbonell and S. Whitaker, *A.I.Ch.E. Symp. Ser. No. 202* **77** (1981).
3. A. B. Eidsath, R. G. Carbonell and S. Whitaker, Dispersion in pulsed systems, *Chem. Engng Sci.* (1982).
4. E. M. Sparrow, B. R. Baliga and S. V. Patankar, Heat transfer and fluid flow analysis of interrupted wall channels, with application to heat exchangers, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **99**, 4–11 (1977).
5. S. V. Patankar, C. H. Liu and E. M. Sparrow, Fully developed flow and heat transfer in ducts having streamwise periodic variations of cross-sectional area, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **99**, 180–186 (1977).
6. S. V. Patankar and C. Prakash, An analysis of the effect of plate thickness on laminar flow and heat transfer in interrupted-plate passages, *Int. J. Heat Mass Transfer* **24**, 1801–1810 (1981).
7. R. Tal (Thau), D. N. Lee and W. A. Sirignano, Hydrodynamics and heat transfer in sphere assemblages: cylindrical cell models, *Int. J. Heat Mass Transfer* **26**, 1265–1273 (1983).
8. L. Hageman and D. Young, *Applied Iterative Methods*. Academic Press, New York (1981).

APPENDIX

ONE-DIMENSIONAL STEADY STATE ANALYSIS OF THE GAS–SOLID HEAT TRANSFER PROBLEM UNDER VARIOUS BOUNDARY CONDITIONS

Assuming only axial variation of the temperature and neglecting diffusive terms, the energy equation on a scale large compared to the distance between spheres is reduced to

$$mC_p \frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{h[T - T_s]S}{2b}.$$

Let

$$\lambda = \frac{hS}{2b} \frac{1}{mC_p},$$

then

$$\frac{\mathrm{d}T}{\mathrm{d}x} = -\lambda[T - T_s].$$

Scaling the axial coordinate

$$z = \lambda x,$$

we get

$$\frac{\mathrm{d}T}{\mathrm{d}Z} = -[T - T_s].$$

For the case that T_s is constant, we define

$$\theta = \frac{T - T_1}{T_s - T_1},$$

and equation (A5) becomes

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = 1 - \theta.$$

As at the inlet ($z = 0$) we have

$$T = T_1, \quad \theta(0) = 0,$$

the solution of equation (A7) is

$$\theta = 1 - e^{-z},$$

or

$$T = T_s + (T_1 - T_s)\,e^{-z},$$

that is to say the temperature difference between the bulk and the wall is decaying exponentially, as suggested by the full two-dimensional analysis.

Int. J. Heat Mass Transfer. Vol. 27, No. 8, pp. 1417–1418, 1984
Printed in Great Britain

0017-9310/84 $3.00 + 0.00
Pergamon Press Ltd.

CONCENTRATION PROFILES NEAR A WALL AT $Pr \gg 1$

B. A. KADER
Institute of Atmospheric Physics, U.S.S.R. Academy of Sciences, Moscow, U.S.S.R.

(*Received* 21 *January* 1983 *and in revised form* 9 *June* 1983)

NOMENCLATURE

c	concentration
d	tube diameter
j	mass flux
Pr	Prandtl number
Re	Reynolds number
u_*	friction velocity
y	coordinate measured normal to a wall.

Greek symbols

δ_h	molecular diffusivity sublayer thickness
λ	skin friction coefficient
ν	kinematic viscosity.

Subscripts

w	wall quantities
o	outer layer quantities
+	dimensionless quantities.

IN REF. [1], an equation has been suggested for the profiles of a passive, i.e. not affecting the dynamics of flow, admixture, which is valid at any value of the molecular Prandtl number. The derivation of this equation was based on interpolation between the limiting laws for temperature or concentration distribution at $Pr \ll 1$ and $Pr \gg 1$. The validity of the law referring to very small Pr numbers is easily verified, since the available literature is abundant in experimental data on temperature profiles in liquid metals where $Pr \ll 1$. The situation with the other limiting case of $Pr \gg 1$ is quite different. The experimental data on heat transfer in liquids of high viscosity are confined to the values of $Pr \lesssim 10^2$, and the accuracy of these data is rather low with the increase of Pr because the heat flux must be very small to justify the treatment of heat as a passive impurity. True, the measurements of concentration profiles of material admixtures in liquids do not require the smallness of the flux and are characterized, as a rule, by $Pr \gtrsim 10^3$, but the data of such measurements have been virtually nonexistent in available literature until recently. The reasons for this are quite obvious: at $Pr \gg 1$ the main concentration drop takes place within a very thin sublayer near a wall, the measurements of which present a tremendous experimental problem. Thus, according to ref. [1], the thickness of the molecular diffusivity sublayer, where more than two-thirds of the total concentration drop is achieved, can be estimated as $\delta_h \approx 12/(Re\,Pr^{1/3}\sqrt{(\lambda/8)})$. Therefore, e.g. in a tube of diameter $d = 20$ mm, δ_h is only about 0.05 mm at $Re = 10^4$ and $Pr = 10^3$.

However, recently data have been published [2] that refer to the measurements of concentration profiles $c(y)$ by the electrochemical method (which does not affect the wall smoothness) at $Pr = 1320$ and $Pr = 2140$ and for the distances y from the wall beginning with y-values of the order of 10 μm. In ref. [2] a photometric method of concentration measurements was used, which is based on the visualization of the surface of fixed concentration with the aid of special additives and colour tracers. It seems worthwhile to compare the general equation proposed in ref. [1] for the evaluation of the concentration profile at any Pr with these new data.

In Fig. 1, the solid line represents this general equation, which in the case of large Pr and $y_+ \gg Pr^{-3}$ takes on the form

$$c^+ = \frac{c - c_w}{c_o - c_w} = 0.068 Pr^{1/3}\, y_+ \exp(-\Gamma) + \exp(-1/\Gamma),$$

$$\Gamma = 2 \times 10^{-3} Pr\, y_+, \; y_+ = y u_*/\nu. \quad (1)$$

The dashed curve corresponds to the formula

$$c^+ = \frac{\sqrt{3}}{2\pi}\left[\frac{1}{2}\ln\frac{(y_+ + A)^2}{y_+^2 - A y_+ + A^2} + \sqrt{3}\arctan\frac{2y_+ - A}{A\sqrt{3}} + \frac{\pi}{2\sqrt{3}}\right], \quad A = (b\,Pr)^{-1/3}, \quad (2)$$

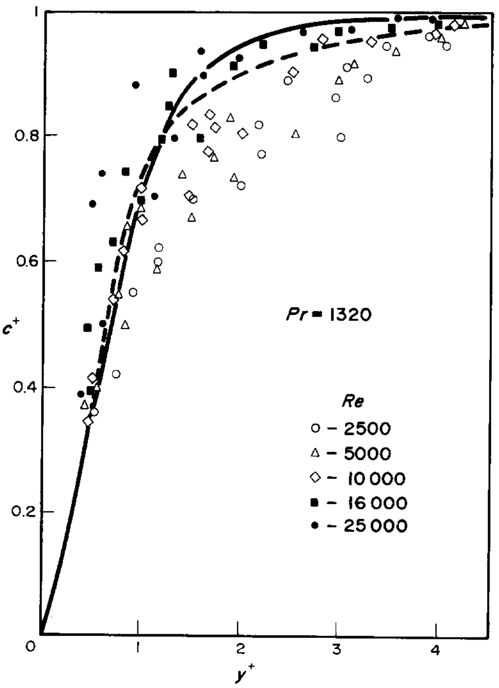

FIG. 1.